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Chiral Solitons With Time-Dependent Coefficients

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Abstract This paper obtains the exact 1-soliton solution to the chiral nonlinear Schrödinger's equation with time-dependent coefficients. Both bright and dark soliton solutions are obtained. The soliton ansatz method is used to carry out the derivation of the soliton.

Keywords Chiral solitons · Time-dependent coefficients

1 Introduction

Chiral solitons appear in the area of Quantum Hall effect where chiral excitations are known to appear. The model that is studied in this context is known as the Jackiw and Pi model [1–10]. This model is given by the chiral nonlinear Schrödinger's equation (NLSE). The interest in this paper is going to be on the chiral NLSE with time-dependent coefficients. The derivation of the soliton solution for the chiral NLSE, with constant coefficients, was carried out by Nishino et al. in 1998 [9]. It is important to consider the time-dependent coefficients to the chiral NLSE as this is more close to reality. It will be seen that the only criterion for the chiral solitons to exist is that the dispersion coefficient must be Riemann integrable.

The chiral NLSE, with time-dependent coefficients, that is going to be studied in this paper is given by

$$iq_t + a(t)q_{xx} + ib(t)\left(qq_x^* - q^*q_x\right)q = 0 \tag{1}$$

In (1), the first term is the evolution term, while $a(t)$ represents the coefficient of dispersion term. Also, $b(t)$ is the coefficient of nonlinear coupling. This nonlinearity is known as the current density. It needs to be noted that (1) is not Galilean invariant and also (1) is not integrable by the method of Inverse Scattering Transform, since it will fail the Painleve test of integrability.

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2 Mathematical Analysis

In order to solve (1), it is assumed that the soliton solution to (1), is given in the form:

$$q(x,t) = P(x,t)e^{i\phi(x,t)} \tag{2}$$

where $P(x,t)$ is the amplitude portion of the soliton, while the phase portion of the soliton is given by

$$\phi(x,t) = \kappa x + \omega t + \theta \tag{3}$$

Here in (3), κ is the frequency of the soliton, ω is the soliton frequency while θ is the phase constant. It needs to be noted that since the coefficients of the dispersion and current density terms in (1) are time-dependent, it is expected that these solitons parameters κ and ω are also consequently time-dependent. Thus from (2) and (3),

$$iq_t = \left\{ i\frac{\partial P}{\partial t} - P\left(\frac{d\kappa}{dt}x + \omega + t\frac{d\omega}{dt} + \frac{d\theta}{dt}\right)\right\} e^{i\phi} \tag{4}$$

$$q_x = \left(\frac{\partial P}{\partial x} + i\kappa P\right) e^{i\phi} \tag{5}$$

$$q_{xx} = \left(\frac{\partial^2 P}{\partial x^2} + 2i\kappa\frac{\partial P}{\partial x} - \kappa^2 P\right) e^{i\phi} \tag{6}$$

so that

$$\left(qq_x^* - q^* q_x\right) q = -2i\kappa P^3 e^{i\phi} \tag{7}$$

Substituting (4), (6) and (7) into (1) and decomposing into real and imaginary parts yields, respectively

$$P\left(\frac{d\kappa}{dt}x + \omega + t\frac{d\omega}{dt} + \frac{d\theta}{dt}\right) - a(t)\left(\frac{\partial^2 P}{\partial x^2} - \kappa^2 P\right) + 2b(t)\kappa P^3 = 0 \tag{8}$$

$$\frac{\partial P}{\partial t} + 2a(t)\kappa\frac{\partial P}{\partial x} = 0 \tag{9}$$

This pair of equations will be analyzed further depending on the type of soliton solution is being fetched. The analysis is now carried out in the following two subsections.

2.1 Bright Solitons

Bright solitons are also known as bell-shaped solitons. These kind of solitons are modeled by the sech function. Therefore, the hypothesis for the function $P(x,t)$ will be given by [2, 9]

$$P(x,t) = \frac{A}{\cosh^p \tau} \tag{10}$$

where

$$\tau = B(x - vt) \tag{11}$$

Here in (10) and (11) A is the soliton amplitude, while B is the inverse width of the soliton and v is the soliton velocity. The index p is unknown at this point and its value will be derived during the course of the derivation of the solution to (1). Again, because of time-dependent coefficients, the soliton parameters A, B and v are all time-dependent. Thus, from the ansatz given by (10), the equation pair (8) and (9) reduces to

$$\frac{A}{\cosh^p \tau}\left(\frac{d\kappa}{dt}x+\omega+t\frac{d\omega}{dt}+\frac{d\theta}{dt}\right) + a(t)\left\{\frac{p^2AB^2}{\cosh^p \tau}-\frac{p(p+1)AB^2}{\cosh^{p+2}\tau}\right\}-\frac{a(t)\kappa^2 A}{\cosh^p \tau}+\frac{2b(t)\kappa A^3}{\cosh^{3p}\tau}=0 \tag{12}$$

and

$$\frac{1}{\cosh^p \tau}\frac{dA}{dt}-\frac{pA\tanh\tau}{\cosh^p\tau}\left\{\frac{\tau}{B}\frac{dB}{dt}-B\left(v+t\frac{dv}{dt}\right)\right\}-\frac{2a(t)\kappa pAB\tanh\tau}{\cosh^p\tau}=0 \tag{13}$$

From (12) setting the exponents $3p$ and $p+2$ equal to one another yields

$$3p=p+2 \tag{14}$$

which gives

$$p=1 \tag{15}$$

Now from (12) noting that $1/\cosh^{p+j}\tau$ are linearly independent functions for $j=0,2$, its coefficients must be respectively set to zero. This leads to

$$\omega=a(t)\left(B^2-\kappa^2\right) \tag{16}$$

and

$$B=A\sqrt{\frac{\kappa b(t)}{a(t)}} \tag{17}$$

which shows that it is necessary to have

$$\kappa a(t)b(t)>0 \tag{18}$$

Also, it is possible to conclude that ω and κ are all constants since their time derivatives are all zero. Now, from (13), the linearly independent functions are $1/\cosh^p\tau$, $\tau\tanh\tau/\cosh^p\tau$ and $\tanh\tau/\cosh^p\tau$. Setting their coefficients respectively to zero, yields

$$\frac{dA}{dt}=0 \tag{19}$$

$$\frac{dB}{dt}=0 \tag{20}$$

and

$$v(t)=\frac{2\kappa}{t}\int a(t)dt \tag{21}$$

so that A and B are constants. Hence, it can be concluded from (17) that it is necessary to have the ratio

$$\frac{a(t)}{b(t)} = \text{constant} \tag{22}$$

for the solitons to exist. Thus, the bright chiral solitons are given by

$$q(x,t) = \frac{A}{\cosh B(x - vt)} e^{i(\kappa x + \omega t + \theta)} \tag{23}$$

where the soliton amplitude A, the with B and the frequency κ are constants while the velocity v and the wave number ω are respectively given by (16) and (21).

2.2 Dark Solitons

The dark solitons are also known as topological solitons or simply topological defects. The hypothesis in this case is given by

$$P(x,t) = A \tanh^p \tau \tag{24}$$

where τ is given in (11). For topological solitons, the parameters A and B are free parameters. Thus, the real and imaginary parts given by (8) and (9) respectively reduce to

$$\begin{aligned} &A \tanh^p \tau \left(\frac{d\kappa}{dt} x + \omega + t\frac{d\omega}{dt} + \frac{d\theta}{dt} \right) - a(t)\kappa^2 A \tanh^p \tau \\ &\quad + a(t)\left\{(p-1)\tanh^{p-2}\tau - 2p\tanh^p \tau + (p+1)\tanh^p \tau\right\} \\ &\quad + 2b(t)\kappa A^3 \tanh^{3p} \tau = 0 \end{aligned} \tag{25}$$

and

$$\begin{aligned} &\frac{dA}{dt}\tanh^p \tau + 2\kappa a(t) pAB \left(\tanh^{p-1}\tau - \tanh^{p+1}\tau\right) \\ &\quad + pA\left(\tanh^{p-1}\tau - \tanh^{p+1}\tau\right)\left\{\frac{\tau}{B} - B\left(v + t\frac{dv}{dt}\right)\right\} = 0 \end{aligned} \tag{26}$$

From (25), setting the exponents $3p$ and $p+2$ leads to the same value of p as in (15). Now, setting the coefficients of the linearly independent functions $\tanh^{p+j}$ to zero for $j = 0, 2$ leads to

$$\omega = -a(t)\left(\kappa^2 + 2B^2\right) \tag{27}$$

and

$$B = A\sqrt{-\frac{\kappa b(t)}{a(t)}} \tag{28}$$

which shows that it is necessary to have

$$\kappa a(t) b(t) < 0 \tag{29}$$

for the dark solitons to exist. Again, similarly, just as in the case of bright solitons, ω and κ are all constants since their time derivatives are all zero. Now, from (26), setting the coefficients of the linearly independent functions $\tanh^p \tau$, $\tau(\tanh^{p-1}\tau - \tanh^{p+1}\tau)$ and $(\tanh^{p-1}\tau - \tanh^{p+1}\tau)$ leads to the same conclusion as in (19) and (20) and (21). Again, since the parameters A and B are constants, the relation (22) also follows from (28). Thus, finally the topological or dark chiral soliton is given by

$$q(x,t) = A\tanh[B(x - vt)]e^{i(\kappa x+\omega t+\theta)} \tag{30}$$

where the relation between the free parameters A and B are given in (28) while the wave number is given by (27). These induce restrictions on the soliton parameters and coefficients that are given by (22) and (29), for dark solitons to exist.

3 Conclusions

This paper obtains the topological and bright soliton solution to the chiral NLSE with time-dependent coefficients. It is only necessary that the time-dependent coefficient of the dispersion term be Riemann integrable as seen from (21) but this function could be otherwise arbitrary. In future, this problem could be extended with perturbation terms, for example with Bohm potential that can be possibly integrated. Another possible extension of this could perhaps be to chiral solitons in $1+2$ dimensions.

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